

# Lifts of partial characters with respect to a chain of normal subgroups

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## Abstract

In this paper, we consider lifts of  $\pi$ -partial characters with the property that the irreducible constituents of their restrictions to certain normal subgroups are also lifts. We will show that such a lift must be induced from what we call an inductive pair, and every character induced from an inductive pair is a such a lift. With this condition, we will get a lower bound on the number of such lifts.

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## 1 Introduction

In his seminal paper [6], Isaacs introduced the idea of  $\pi$ -partial characters that for  $\pi$ -separable groups are an analog of  $p$ -Brauer characters for  $p$ -solvable groups. Another approach to  $\pi$ -partial characters, which is the one we will take, can be found in [8].

Let  $\pi$  be a set of primes and let  $G$  be a  $\pi$ -separable group. Take  $G^0$  to be the set of  $\pi$ -elements in  $G$ . The  $\pi$ -partial characters of  $G$  are to defined to be the restrictions to  $G^0$  of all of the characters of  $G$ . A  $\pi$ -partial character is called irreducible if it cannot be written as the sum of other  $\pi$ -partial characters, and we write  $I_\pi(G)$  for the set of irreducible  $\pi$ -partial characters of  $G$ .

If  $\chi$  is a character of  $G$ , then we write  $\chi^0$  for the restriction of  $\chi$  to  $G^0$ . When  $\varphi$  is a  $\pi$ -partial character of  $G$ , we say that  $\chi$  is a *lift* of  $\varphi$  if  $\chi^0 = \varphi$ . In general, we say  $\chi$  is a  $\pi$ -lift if  $\chi^0 \in I_\pi(G)$ . Because of the way we constructed the  $\pi$ -characters, we know that all  $\pi$ -partial characters have  $\pi$ -lifts. One main thread of research in this subject has been to find sets of canonical lifts for  $I_\pi(G)$ . A set of  $\pi$ -lifts  $\mathcal{A}$  is called canonical if restriction is a bijection from  $\mathcal{A}$  to  $I_\pi(G)$ , and there is some canonical property used to determine  $\mathcal{A}$ . The prototypical example of a canonical set of  $\pi$ -lifts is the set  $B_\pi(G)$  that was defined in [6] by Isaacs. If 2 is not in  $\pi$ , then using a suggestion of Dade, Isaacs has constructed in [7] a

second set of canonical  $\pi$ -lifts  $D_\pi(G)$ . In our papers [10] and [12], we discuss two other ways to construct sets of canonical  $\pi$ -lifts.

Recently, J. P. Cossey has looked at a different question regarding lifts. In particular, he has focused on a single irreducible  $\pi$ -partial character  $\varphi$ , and he has studied the set of all lifts of  $\varphi$  in [1], [2], and [3]. In so doing, he has asked a number of questions regarding the lifts of  $\varphi$ . One question he has asked in a talk at the 2007 Zassenhaus group theory conference (see [4]) is whether we can characterize all the lifts of  $\varphi$ , and in particular, whether there is some character pair from which all the lifts of  $\varphi$  are induced. One purpose of this note is to present an example that shows that no such pair needs to exist.

However, we actually wish to do more. We wish to study further the question of characterizing all of the lifts of a  $\pi$ -partial character. At this time, the question of characterizing all of the lifts of  $\varphi$  seems out of reach. Thus, it seems reasonable as a first step to add additional conditions that the lifts must satisfy and then characterize the lifts that satisfy these additional conditions. For example, one condition that it makes sense to look at are lifts of  $\varphi$  whose restrictions to every normal subgroup have irreducible constituents that are  $\pi$ -lifts. The  $\pi$ -lifts in  $B_\pi(G)$  and  $D_\pi(G)$  have this property. It seems reasonable to ask if there are any other  $\pi$ -lifts with this property.

Rather than look at all normal subgroups at once, we restrict our attention to a single normal series. A *normal  $\pi$ -series* for  $G$  is a set  $\mathcal{N} = \{1 = N_0, N_1, \dots, N_n = G\}$  of normal subgroups in  $G$  where  $N_i \leq N_{i+1}$  and  $N_{i+1}/N_i$  is either a  $\pi$ -group or a  $\pi'$ -group for each  $i$ . We say  $\chi$  is an  $\mathcal{N}$ - $\pi$ -lift if for each  $N \in \mathcal{N}$ , the irreducible constituents of  $\chi_N$  are  $\pi$ -lifts. In particular, since  $G \in \mathcal{N}$ , this requires that  $\chi$  be a  $\pi$ -lift.

In this note, we characterize the  $\mathcal{N}$ - $\pi$ -lifts of  $\varphi$ . Notice that  $\chi$  is an  $\mathcal{N}$ -lift for all  $\pi$ -series  $\mathcal{N}$  if and only if the irreducible constituents of  $\chi_N$  are  $\pi$ -lifts for every normal subgroup  $N$  of  $G$ , and so, by characterizing the  $\mathcal{N}$ - $\pi$ -lifts for all  $\mathcal{N}$ , we obtain a characterization of the lifts whose restrictions to normal subgroups have irreducible constituents that are  $\pi$ -lifts. In [10] and [12], we discussed two ways of constructing canonical  $\pi$ -lifts based on  $\mathcal{N}$ . We will use  $B_\pi(G : \mathcal{N})$  to denote the canonical  $\pi$ -lifts developed in [10] and  $B_\pi(G : *\mathcal{N})$  for the  $\pi$ -lifts found in [12].

To do this, we will use some tools developed in [10]. Let  $G$  be a  $\pi$ -separable group and  $\mathcal{N}$  a  $\pi$ -series for  $G$ . For every character  $\chi \in \text{Irr}(G)$ , we associated in [10] a character pair  $(T, \tau)$  which we called a *self-stabilizing pair* with respect to  $\mathcal{N}$ . (A character pair  $(T, \tau)$  is a subgroup  $T \leq G$  and character  $\tau \in \text{Irr}(T)$ . The self-stabilizing pair is well-defined only up to conjugacy.) This pair has the properties of a nucleus in that  $\tau^G = \chi$  and  $\tau$  factors into a product of a  $\pi$ -special character and a  $\pi'$ -special character. In this paper, we will define what it means for a pair  $(V, \gamma)$  to be *inductive* for  $\mathcal{N}$ . With this definition, we determine which characters in  $\text{Irr}(G)$  are  $\mathcal{N}$ - $\pi$ -lifts.

**Theorem 1.** *Let  $\pi$  be a set of primes, let  $G$  be a  $\pi$ -separable group, and  $\mathcal{N}$  a normal  $\pi$ -series for  $G$ . Consider a character  $\chi \in \text{Irr}(G)$  with self-stabilizing pair  $(V, \gamma)$  with respect to  $\mathcal{N}$ . Assume that  $\gamma = \alpha\beta$  where  $\alpha$  is  $\pi$ -special and  $\beta$  is  $\pi'$ -special. Then the following are equivalent:*

1.  $\chi$  is an  $\mathcal{N}$ - $\pi$ -lift.
2.  $(V, \gamma)$  is inductive with respect to  $\mathcal{N}$ .

3.  $(V, \alpha)$  is inductive with respect to  $\mathcal{N}$  and  $\beta$  is linear.

We can use this characterization of the  $\mathcal{N}$ - $\pi$ -lifts to obtain a lower bound on the number of  $\mathcal{N}$ - $\pi$ -lifts.

**Theorem 2.** *Let  $\pi$  be a set of primes, let  $G$  be a  $\pi$ -separable group, and  $\mathcal{N}$  a normal  $\pi$ -series for  $G$ . Fix  $\pi$ -partial character  $\varphi \in \mathbf{I}_\pi(G)$  and character  $\chi \in \mathbf{B}_\pi(G : \mathcal{N})$  so that  $\chi^0 = \varphi$ . Let  $(V, \gamma)$  be a self-stabilizing pair for  $\chi$ . Then the following are true.*

1. *There is an injective map from the linear characters of  $V/V'$  with  $\pi'$ -order to the  $\mathcal{N}$ - $\pi$ -lifts of  $\varphi$ .*
2. *The number of  $\mathcal{N}$ - $\pi$ -lifts for  $\varphi$  is at least  $|V : V'|_{\pi'}$ .*

We return to our initial question of classifying the  $\pi$ -lifts with the property that the restriction to every normal subgroup has constituents that are lifts. These  $\pi$ -lifts can now be classified since they are precisely the  $\pi$ -lifts that are  $\mathcal{N}$ - $\pi$ -lifts for all normal  $\pi$ -series  $\mathcal{N}$ . Unfortunately, it is not clear whether this classification is practical.

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## 2 Characterizing $\mathcal{N}$ -lifts

For now, we will let  $\mathcal{N}$  be a collection of normal subgroups of  $G$ . We say  $\mathcal{L}$  is a *compatible* set of  $\pi$ -lifts for  $\mathcal{N}$  if the following hold:

1. For each  $N \in \mathcal{N}$ , we have  $\mathcal{L}(N) \subseteq \text{Irr}(N)$  and  $\alpha \mapsto \alpha^0$  is bijection from  $\mathcal{L}(N)$  to  $\mathbf{I}_\pi(N)$ .
2. If  $M \leq N$  with  $M, N \in \mathcal{N}$ , then the irreducible constituents of  $\alpha_M$  lie in  $\mathcal{L}(M)$  for all  $\alpha \in \mathcal{L}(N)$ .

In [6], Isaacs proved that if  $\chi \in \mathbf{B}_\pi(G)$  and  $N$  is a normal subgroup of  $G$ , then the irreducible constituents of  $\chi_N$  lie in  $\mathbf{B}_\pi(N)$ . It follows that  $\mathbf{B}_\pi(\cdot)$  is a compatible set of  $\pi$ -lifts for all  $\mathcal{N}$ . Similarly, when  $2 \notin \pi$ , Isaacs proved in [7] if  $\chi \in \mathbf{D}_\pi(G)$  and  $N$  is any normal subgroup that the irreducible constituents of  $\chi_N$  lie in  $\mathbf{D}_\pi(N)$ , and so,  $\chi$  will be a  $\mathcal{N}$ - $\pi$ -lift for all  $\mathcal{N}$ . Hence,  $\mathbf{D}_\pi(\cdot)$  is a compatible set of  $\pi$ -lifts for all  $\mathcal{N}$ . The results proved in [10] and [11] imply that both  $\mathbf{B}_\pi(\cdot : \mathcal{N})$  and  $\mathbf{B}_\pi(\cdot : *\mathcal{N})$  are both compatible sets of  $\pi$ -lifts for  $\mathcal{N}$  when  $\mathcal{N}$  is a normal  $\pi$ -series. We now show that compatible sets of  $\pi$ -lifts are compatible with conjugation.

**Lemma 2.1.** *Let  $\pi$  be a set of primes, let  $G$  be a  $\pi$ -separable group, let  $\mathcal{N}$  be a collection of normal subgroups of  $G$ , and let  $\mathcal{L}$  be a compatible set of  $\pi$ -lifts for  $\mathcal{N}$ . If  $N \in \mathcal{N}$  and  $\alpha \in \mathcal{L}(N)$ , then  $\alpha^g \in \mathcal{L}(N)$  for all  $g \in G$ .*

*Proof.* We know that  $\alpha^0 \in \mathbf{I}_\pi(N)$ . Let  $\varphi \in \mathbf{I}_\pi(G)$  so that  $\alpha^0$  is a constituent of  $\varphi_N$ . Take  $\chi \in \mathcal{L}(G)$  so that  $\chi^0 = \varphi$ . Now,  $\alpha^0$  is an irreducible constituent of  $\varphi_N = (\chi^0)_N = (\chi_N)^0$ . Since every irreducible constituent of  $\chi_N$  lies in  $\mathcal{L}(N)$  and  $\alpha$  is the unique element of  $\mathcal{L}(N)$  that is a  $\pi$ -lift of  $\alpha^0$ , we deduce that  $\alpha$  is a constituent of  $\chi_N$ . Also,  $\alpha^g$  is a constituent of  $\chi_N$ , so  $\alpha^g \in \mathcal{L}(N)$  for all  $g \in G$ .  $\square$

One of the motivations for looking at compatible sets of lifts is that if we have a  $\pi$ -partial character of a normal subgroup in the collection, then its compatible lift will have the same stabilizer in  $G$ .

**Corollary 2.2.** *Let  $\pi$  be a set of prime, let  $G$  be a  $\pi$ -separable group, let  $\mathcal{N}$  be a collection of normal subgroups of  $G$ , and let  $\mathcal{L}$  be a compatible set of  $\pi$ -lifts for  $\mathcal{N}$ . If  $N \in \mathcal{N}$  and  $\alpha \in \mathcal{L}(N)$ , then  $G_\alpha = G_{\alpha^0}$ .*

*Proof.* It is obvious that if  $g$  stabilizes  $\alpha$ , then it stabilizes  $\alpha^0$ , so  $G_\alpha \leq G_{\alpha^0}$ . Suppose now that  $g$  stabilizes  $\alpha^0$ . By Lemma 2.1, we have that  $\alpha^g \in \mathcal{L}(N)$ . Now,  $(\alpha^g)^0 = (\alpha^0)^g = \alpha^0$ . Since restriction is a bijection from  $\mathcal{L}(N)$  to  $\text{I}_\pi(N)$ , it follows that  $\alpha^g = \alpha$ . We conclude that  $G_\alpha = G_{\alpha^0}$ .  $\square$

We will now specialize to the case where  $\mathcal{N}$  is a normal  $\pi$ -series. At this time, we need to review the definition and results regarding self-stabilizing pairs that appeared in [10]. Let  $\mathcal{N} = \{1 = N_0 \leq N_1 \leq \dots \leq N_n = G\}$  be a normal  $\pi$ -series for  $G$ . Fix  $\chi \in \text{Irr}(G)$ . A *character tower* for  $\chi$  with respect to  $\mathcal{N}$  is a set of characters  $\mathcal{U} = \{1 = \nu_0, \nu_1, \dots, \nu_n = \chi\}$  where each  $\nu_i \in \text{Irr}(N_i)$  and  $\nu_i$  is a constituent of  $(\nu_{i+1})_{N_i}$  for all  $i = 0, \dots, n-1$ . We take  $T$  to be  $G_{\mathcal{U}}$ , the stabilizer in  $G$  of  $\mathcal{U}$ . We proved in [10] that there is a unique character  $\tau \in \text{Irr}(T)$  so that  $\tau^G = \chi$  and  $\tau_{T \cap N_i} = t_i \tau_i$  for some character  $\tau_i \in \text{Irr}(T \cap N_i)$  with  $(\tau_i)^{N_i} = \nu_i$ . We say that  $(T, \tau)$  is the *self-stabilizing pair* determined by  $\mathcal{U}$ . It is proved in [10] that all the self-stabilizing pairs determined by character towers for  $\chi$  are conjugate in  $G$ . We also prove that  $\tau$  must be  $\pi$ -factored. We defined  $\text{B}_\pi(G : \mathcal{N})$  to be the characters in  $\text{Irr}(G)$  who have a self-stabilizing pair  $(T, \tau)$  where  $\tau$  is  $\pi$ -special.

We continue to suppose that  $\varphi \in \text{I}_\pi(G)$ . We now show that if  $\psi \in \mathcal{L}(G)$  with  $\psi^0 = \varphi$  and  $\chi$  is any  $\mathcal{N}$ - $\pi$ -lift of  $\varphi$ , then the subgroup for a self-stabilizing pair for  $\chi$  is contained in the subgroup of a self-stabilizing pair for  $\psi$ .

**Lemma 2.3.** *Let  $\pi$  be a set of primes, let  $G$  be a  $\pi$ -separable group, let  $\mathcal{N}$  be a normal  $\pi$ -series for  $G$ , and let  $\mathcal{L}$  be a compatible set of  $\pi$ -lifts for  $\mathcal{N}$ . Assume  $\varphi$  is a  $\pi$ -partial character of  $G$ , and fix  $\psi \in \mathcal{L}(G)$  so that  $\psi^0 = \varphi$ . Suppose  $\chi$  is an  $\mathcal{N}$ - $\pi$ -lift of  $\varphi$ . Let  $\mathcal{T} = \{1 = \tau_0, \tau_1, \dots, \tau_n = \psi\}$  and  $\mathcal{U} = \{1 = \nu_0, \nu_1, \dots, \nu_n = \chi\}$  be character towers with respect to  $\mathcal{N}$  for  $\psi$  and  $\chi$  respectively so that  $\tau_i^0 = \nu_i^0$  for all  $i$ . Then  $G_{\mathcal{U}} \leq G_{\mathcal{T}}$ .*

*Proof.* If  $x \in G_{\mathcal{U}}$ , then  $\nu_i^x = \nu_i$  for each  $i$ . This implies that  $(\nu_i^x)^0 = \nu_i^0 = \tau_i^0$  for each  $i$ . On the other hand, if  $g \in G^0$ , then  $(\nu_i^x)^0(g) = (\nu_i)(g^{x^{-1}}) = (\tau_i)(g^{x^{-1}}) = (\tau_i^x)^0(g)$ , and this implies that  $\tau_i^0 = (\tau_i^x)^0$  for each  $i$ . We now note that  $\tau_i, \tau_i^x \in \mathcal{L}(N_i)$  using Lemma 2.1. Since restriction to  $G^0$  is a bijection from  $\mathcal{L}(N_i)$  to  $\text{I}_\pi(N_i)$ , this implies that  $\tau_i = \tau_i^x$  for all  $i$ . We conclude that  $x \in G_{\mathcal{T}}$  as desired.  $\square$

We now show that if  $\chi \in \mathcal{L}(G)$  and  $(T, \tau)$  is a self-stabilizing pair for  $\chi$  with respect to  $\mathcal{N}$ , then there is a  $\pi$ -special character  $\sigma \in \text{Irr}(T)$  so that  $(T, \sigma)$  is a self-stabilizing pair for  $\mathcal{N}$  and  $(\sigma^G)^0 = \chi^0$ .

**Lemma 2.4.** *Let  $\pi$  be a set of primes, let  $G$  be a  $\pi$ -separable group, let  $\mathcal{N}$  be a normal  $\pi$ -series for  $G$ , and let  $\mathcal{L}$  be a compatible set of  $\pi$ -lifts for  $\mathcal{N}$ . Suppose  $\chi \in \mathcal{L}(G)$  and  $(T, \tau)$  is a self-stabilizing pair for  $\chi$  with respect to  $\mathcal{N}$ . Then  $\tau(1)$  is a  $\pi$ -number. Furthermore, there is a character  $\sigma \in \text{Irr}(T)$  so that  $\sigma$  is  $\pi$ -special,  $(T, \sigma)$  is a self-stabilizing pair with respect to  $\mathcal{N}$ , and  $(\sigma^G)^0 = \chi^0$ .*

*Proof.* Label  $\mathcal{N} = \{1 = N_0 \leq N_1 \leq \dots \leq N_n = G\}$ , and let  $\mathcal{T} = \{1 = \theta_0, \theta_1, \dots, \theta_n = \chi\}$  be the character tower under  $\chi$  that yields  $(T, \tau)$ . We know that  $\theta_i \in \mathcal{L}(N_i)$  for each  $i$ , and so,  $\varphi_i = \theta_i^0 \in \text{I}_\pi(N_i)$ . Recall that  $T = G_{\mathcal{T}}$ .

We showed in [10] that restriction is a bijection from  $\text{B}_\pi(G : \mathcal{N})$  to  $\text{I}_\pi(G)$ . Since  $\chi^0 \in \text{I}_\pi(G)$ , there is a character  $\psi \in \text{B}_\pi(G : \mathcal{N})$  so that  $\psi^0 = \chi^0$ . Let  $\mathcal{N}_i = \{N_0 \leq N_1 \leq \dots \leq N_i\}$ . Also, it was proved in Theorem 5.2 of [10] that all the irreducible constituents of  $\psi_{N_i}$  lie in  $\text{B}_\pi(N_i : \mathcal{N}_i)$ . Let  $\gamma_i$  be the unique character in  $\text{B}_\pi(N_i : \mathcal{N}_i)$  such that  $\gamma_i^0 = \varphi_i$ . Since  $\varphi_i$  is a constituent of  $(\psi^0)_{N_i} = (\chi^0)_{N_i}$ , it follows that  $\gamma_i$  is a constituent of  $\psi_{N_i}$ . In fact, if  $j \geq i$ , then  $\gamma_i$  will be a constituent of  $(\gamma_j)_{N_i}$ . We conclude that  $\mathcal{U} = \{\gamma_0, \gamma_1, \dots, \gamma_n\}$  is a character tower for  $\mathcal{N}$  under  $\psi$ . Let  $(S, \sigma)$  be the self-stabilizing pair determined by this character tower, and recall that  $S = G_{\mathcal{U}}$ . Since  $\chi \in \mathcal{L}(G)$ , we have  $S \leq T$  by Lemma 2.3. On the other hand, since  $\text{B}_\pi(\cdot : \mathcal{N})$  is a compatible set of  $\pi$ -lifts for  $\mathcal{N}$ , we obtain  $T \leq S$  by Lemma 2.3. We conclude that  $S = T$ . We now have  $|G : T|\tau(1) = \chi(1) = \psi(1) = |G : T|\sigma(1)$ , and  $\tau(1) = \sigma(1)$ . Since  $\psi \in \text{B}_\pi(G : \mathcal{N})$ , we know that  $\sigma$  is  $\pi$ -special, and therefore,  $\sigma(1)$  is a  $\pi$ -number.  $\square$

For our characterization, we need one further definition. The character pair  $(V, \gamma)$  is called *inductive for  $\mathcal{N}$*  if for each  $N \in \mathcal{N}$ , we have  $\gamma_{V \cap N} = a\eta$  for some positive integer  $a$  and character  $\eta \in \text{Irr}(V \cap N)$  where  $(\eta^N)^0$  is irreducible. This gives one implication of Theorem 1.

**Lemma 2.5.** *Let  $\pi$  be a set of primes, let  $G$  be a  $\pi$ -separable group, and let  $\mathcal{N}$  be a normal  $\pi$ -series for  $G$ . Let  $(V, \gamma)$  be inductive for  $\mathcal{N}$ . Then  $\gamma^G \in \text{Irr}(G)$  is an  $\mathcal{N}$ - $\pi$ -lift and  $\gamma$  is  $\pi$ -factored.*

*Proof.* Since  $G \in \mathcal{N}$ , we have  $(\gamma^G)^0$  is irreducible, and thus,  $\gamma^G \in \text{Irr}(G)$  will be a  $\pi$ -lift. For each  $N \in \mathcal{N}$ , we have that  $\gamma_{V \cap N} = a\eta$  for a positive integer  $a$  and character  $\eta \in \text{Irr}(V \cap N)$ . We know that  $(\eta^N)^0$  is irreducible. Notice that  $\eta^N$  is an irreducible constituent of  $\chi_N$ . Since the irreducible constituents of  $\chi_N$  are all conjugate, this implies that the irreducible constituents of  $\chi_N$  are  $\pi$ -lifts, and hence,  $\chi$  is an  $\mathcal{N}$ - $\pi$ -lift. The fact that  $\gamma$  is  $\pi$ -factored follows from Theorem 21.7 of [13].  $\square$

The next lemma is useful for identifying pairs that are inductive for  $\mathcal{N}$ . This gives another implication of Theorem 1.

**Lemma 2.6.** *Let  $\pi$  be a set of primes, let  $G$  be a  $\pi$ -separable group, and let  $\mathcal{N}$  be a normal  $\pi$ -series for  $G$ . Let  $(V, \gamma)$  be a pair so that  $\gamma$  factors as  $\alpha\beta$  where  $\alpha$  is  $\pi$ -special and  $\beta$  is  $\pi'$ -special. If  $\beta$  is linear, then  $(V, \alpha)$  is inductive for  $\mathcal{N}$  if and only if  $(V, \gamma)$  is inductive for  $\mathcal{N}$ .*

*Proof.* Suppose first that  $(V, \gamma)$  is inductive for  $\mathcal{N}$ . For each  $N \in \mathcal{N}$ , it follows that  $\gamma_{V \cap N} = (\alpha\beta)_{V \cap N}$  is homogeneous. We deduce that  $\alpha_{V \cap N} = e\eta$  for some  $\pi$ -special character  $\eta$  of  $V \cap N$  and positive integer  $e$ . Since  $\beta$  is linear and  $\pi'$ -special, we have  $\beta_{V \cap N} = \xi$  where  $\xi$  is linear and  $\pi'$ -special. Hence, we have  $(\alpha\beta)_{V \cap N} = e\eta\xi$ . Because  $(V, \gamma)$  is inductive for  $\mathcal{N}$ , we know that  $((\eta\xi)^N)^0$  is irreducible. Also,  $(\eta\xi)^0 = \eta^0$  as  $\xi$  is linear and  $\pi'$ -special, so  $(\eta^N)^0 = ((\eta\xi)^N)^0$  is irreducible. We conclude that  $(V, \alpha)$  is inductive for  $\mathcal{N}$  as desired.

We now suppose that  $(V, \alpha)$  is inductive for  $\mathcal{N}$ . Thus, for  $N \in \mathcal{N}$ , we have that  $\alpha_{V \cap N} = a\eta$  for a positive integer  $a$  and character  $\eta \in \text{Irr}(V \cap N)$  and  $(\eta^N)^0$  is irreducible. Since  $\beta$  is linear and  $\pi'$ -special,  $\beta_N = \xi$  is irreducible and  $\pi'$ -special. It follows that  $\gamma_{V \cap N} = a\eta\xi$  and  $(\eta\xi)^0 = \eta^0$ . We deduce that  $((\eta\xi)^N)^0 = (\eta^N)^0$  is irreducible. This implies that  $(V, \gamma)$  is inductive.  $\square$

We have shown that if  $\chi$  is an induced from a pair that is inductive with respect to  $\mathcal{N}$ , then  $\chi$  is an  $\mathcal{N}$ - $\pi$ -lift. We now show that if  $\chi$  is an  $\mathcal{N}$ - $\pi$ -lift, then such a pair must exist. We now prove Theorem 1 which shows that  $\chi$  is an  $\mathcal{N}$ - $\pi$ -lift if and only if a self-stabilizing pair  $(V, \gamma)$  for  $\chi$  with respect to  $\mathcal{N}$  is inductive for  $\mathcal{N}$ . The more interesting portion of this theorem considers the  $\pi$ -factorization of  $\gamma$ . In this case, we show that the  $\pi'$ -special factor of  $\gamma$  must be linear, and the  $\pi$ -special factor will afford an inductive pair for  $\mathcal{N}$ .

*Proof of Theorem 1.* Suppose first that  $\chi$  is an  $\mathcal{N}$ - $\pi$ -lift. We show that  $(V, \alpha)$  is inductive. Write  $\mathcal{N} = \{1 = N_0, N_1, \dots, N_n = G\}$ , and let  $\mathcal{U} = \{1 = \nu_0, \nu_1, \dots, \nu_n = \chi\}$  be the character tower with respect to  $\mathcal{N}$  affording  $(V, \alpha\beta)$ . We can find  $\mu_i \in B_\pi(N_i)$  so that  $\mu_i^0 = \nu_i^0$ . Set  $\psi = \mu_n$  and  $\mathcal{T} = \{\mu_0, \mu_1, \dots, \mu_n = \psi\}$ . Notice that  $\psi^0 = \chi^0$ , and  $\mathcal{T}$  is character tower for  $\psi$  with respect to  $\mathcal{N}$ . Let  $(T, \tau)$  be the self-stabilizing pair afforded by  $\mathcal{T}$ . By Lemma 2.3, we see that  $V \leq T$ , and by Lemma 2.4, we know that  $\tau(1)$  is a  $\pi$ -number. We have  $\chi(1) = |G : V|\alpha(1)\beta(1)$  and  $\psi(1) = |G : T|\tau(1)$ . Since  $\chi(1) = \psi(1)$ , this implies that  $|G : V|\alpha(1)\beta(1) = |G : T|\tau(1)$ , and hence,  $|T : V|\alpha(1)\beta(1) = \tau(1)$ . Now,  $\beta(1)$  divides the  $\pi$ -number  $\tau(1)$ . On the other hand,  $\beta$  is  $\pi'$ -special, so  $\beta(1)$  is a  $\pi'$ -number. Together, these imply that  $\beta(1) = 1$ , and so,  $\beta$  is linear.

For each  $N \in \mathcal{N}$ , since  $(V, \alpha\beta)$  is a self-stabilizing pair, it follows that  $(\alpha\beta)_{V \cap N}$  is homogeneous. We deduce that  $\alpha_{V \cap N} = e\eta$  for some  $\pi$ -special character  $\eta$  of  $V \cap N$  and positive integer  $e$ . Since  $\beta$  is linear, we have  $\beta_{V \cap N} = \xi$  where  $\xi$  is linear. Hence, we have  $(\alpha\beta)_{V \cap N} = e\eta\xi$ . Because  $(V, \alpha\beta)$  is self-stabilizing for  $\chi$ , we know that  $\nu = (\eta\xi)^N$  is an irreducible constituent of  $\chi_N$ . Since  $\chi$  is an  $\mathcal{N}$ - $\pi$ -lift, this implies that  $\nu^0$  is irreducible. Also,  $(\eta\xi)^0 = \eta^0$  as  $\xi$  is linear and  $\pi'$ -special, so  $(\eta^N)^0 = ((\eta\xi)^N)^0 = \nu^0$  is irreducible. We conclude that  $(V, \alpha)$  is inductive for  $\mathcal{N}$  as desired.

If  $(V, \alpha)$  is inductive for  $\mathcal{N}$  and  $\beta$  is linear, then Lemma 2.6 implies that  $(V, \gamma)$  is inductive for  $\mathcal{N}$ . Finally, if  $(V, \gamma)$  is inductive for  $\mathcal{N}$ , then Lemma 2.5 implies that  $\chi$  is an  $\mathcal{N}$ - $\pi$ -lift.  $\square$

### 3 Counting $\mathcal{N}$ -lifts

We now turn our attention to the lower bound on the number of  $\mathcal{N}$ - $\pi$ -lifts which is proved in Theorem 2. To do this, we need to gather more results regarding inductive pairs. We now show that if we have a pair that is inductive, then the subgroup of that pair must be contained in the subgroup for some self-stabilizing pair.

**Lemma 3.1.** *Let  $\pi$  be a set of primes, let  $G$  be a  $\pi$ -separable group, and let  $\mathcal{N}$  be a normal  $\pi$ -series for  $G$ . Let  $(V, \gamma)$  be inductive for  $\mathcal{N}$ , and write  $\chi = \gamma^G \in \text{Irr}(G)$ . Then  $\chi$  has a self-stabilizing pair  $(U, \delta)$  with respect to  $\mathcal{N}$  so that  $V \leq U$ .*

*Proof.* Write  $\mathcal{N} = \{1 = N_0 \leq N_1 \leq \dots \leq N_n = G\}$ . For each  $i$ , we have  $\gamma_{N_i} = a_i\eta_i$  for some positive integer  $a_i$  and character  $\eta_i \in \text{Irr}(V \cap N_i)$ . Also, we have  $\nu_i = (\eta_i)^{N_i}$  satisfies

$(\nu_i)^0$  is irreducible and so,  $\nu_i \in \text{Irr}(N_i)$ . It is not difficult to see that  $\mathcal{U} = \{\nu_0, \nu_1, \dots, \nu_n\}$  is character tower for  $\chi$  with respect to  $\mathcal{N}$ . Let  $(U, \delta)$  be the self-stabilizing pair afforded by  $\mathcal{U}$ . Since  $V$  stabilizes each  $\eta_i$ , we see that  $V$  stabilizes each  $\nu_i$ , and so,  $V$  stabilizes  $\mathcal{U}$ . This implies that  $V \leq U$ .  $\square$

This yields the following useful observation.

**Corollary 3.2.** *Let  $\pi$  be a set of primes, let  $G$  be a  $\pi$ -separable group, and let  $\mathcal{N}$  be a normal  $\pi$ -series for  $G$ . Let  $(V, \gamma)$  be a pair so that  $\gamma$  factors as  $\alpha\beta$  where  $\alpha$  is  $\pi$ -special and  $\beta$  is  $\pi'$ -special. Then  $(V, \gamma)$  is inductive for  $\mathcal{N}$  if and only if  $\beta$  is linear and  $(V, \alpha)$  is inductive for  $\mathcal{N}$ .*

*Proof.* Suppose first that  $\beta$  is linear and  $(V, \alpha)$  is inductive for  $\mathcal{N}$ . Then  $(V, \gamma)$  is inductive for  $\mathcal{N}$  by Lemma 2.6.

Conversely, we suppose that  $(V, \gamma)$  is inductive. By Lemma 2.5, we know that  $\chi = \gamma^G \in \text{Irr}(G)$  is an  $\mathcal{N}$ - $\pi$ -lift. From Lemma 3.1,  $\chi$  has a self-stabilizing pair  $(U, \delta)$  so that  $V \leq U$ . By Lemma 2.4, we know that  $\delta(1)$  is a  $\pi$ -number. We have  $|G : U|\delta(1) = \chi(1) = |G : V|\alpha(1)\beta(1)$ , and hence,  $\delta(1) = |U : V|\alpha(1)\beta(1)$ . Thus,  $\beta(1)$  divides  $\delta(1)$ . Since  $\delta(1)$  is a  $\pi$ -number and  $\beta$  is  $\pi'$ -special, we conclude that  $\beta(1) = 1$ . We now apply Lemma 2.6 to see that  $(V, \alpha)$  is inductive for  $\mathcal{N}$ .  $\square$

If  $(V, \alpha)$  is a self-stabilizing pair for  $G$  with respect to  $\mathcal{N}$  where  $\alpha$  is  $\pi$ -special, then we show that  $(V, \alpha)$  is inductive for  $\mathcal{N}$ .

**Corollary 3.3.** *Let  $\pi$  be a set of primes, let  $G$  be a  $\pi$ -separable group, and let  $\mathcal{N}$  be a normal  $\pi$ -series for  $G$ . Suppose that  $(V, \alpha)$  is a self-stabilizing pair with respect to  $\mathcal{N}$ . If  $\alpha$  is  $\pi$ -special, then  $(V, \alpha)$  is inductive for  $\mathcal{N}$ .*

*Proof.* Let  $\chi \in \text{Irr}(G)$  be the character associated with  $(V, \alpha)$ . We proved in Theorem B and Theorem 5.2 of [10] that  $\chi$  is an  $\mathcal{N}$ - $\pi$ -lift. Applying Theorem 1, we conclude that  $(V, \alpha)$  is inductive for  $\mathcal{N}$ .  $\square$

Finally, we have a condition which will be used to find other self-stabilizers pair using a given self-stabilizing pair.

**Lemma 3.4.** *Let  $\pi$  be a set of primes, let  $G$  be a  $\pi$ -separable group, and let  $\mathcal{N}$  be a normal  $\pi$ -series for  $G$ . Suppose that  $(V, \alpha)$  is a self-stabilizing pair with respect to  $\mathcal{N}$  where  $\alpha$  is  $\pi$ -special. Assume  $\beta \in \text{Irr}(V)$  is  $\pi'$ -special. If  $\beta_{V \cap N}$  is homogeneous for all  $N \in \mathcal{N}$ , then  $(V, \alpha\beta)$  is a self-stabilizing pair with respect to  $\mathcal{N}$ .*

*Proof.* We work by induction on  $|G|$ . Let  $N$  be the penultimate term of  $\mathcal{N}$ . Let  $\alpha_{V \cap N} = a\eta$  and  $\beta_{V \cap N} = b\xi$ . Write  $\mathcal{N}^*$  for the normal  $\pi$ -series by terminating  $\mathcal{N}$  at  $N$ . We know that  $(V \cap N, \eta)$  is a self-stabilizing pair with respect to  $\mathcal{N}^*$  and that its stabilizer in  $G$  is  $V$ . By induction,  $(V \cap N, \eta\xi)$  is a self-stabilizing pair with respect to  $\mathcal{N}^*$ . Obviously,  $\eta\xi$  is invariant in  $V$  and since  $V$  is the stabilizer of  $(V \cap N, \eta)$ , it follows that it stabilizes  $(V \cap N, \eta\xi)$ . We conclude that  $(V, \alpha\beta)$  is a self-stabilizing pair for  $\mathcal{N}$ .  $\square$

For the next proof, we need the theory of source maps that we developed in [12]. Following [5] and [9], we say that a character pair  $(H, \varphi)$  is an *inductive source* if induction is an injection from  $\text{Irr}(T \mid \varphi)$  to  $\text{Irr}(G)$  where  $T$  is the stabilizer of  $(H, \varphi)$  in  $G$ . Let  $G$  be a finite group, and let  $\text{pair}(G)$  be the set of all character pairs in  $G$ . We define a *source map* on  $G$  to be a function  $\mathcal{M} : \text{pair}(G) \rightarrow \text{pair}(G)$  that satisfies for all pairs  $(H, \theta) \in \text{pair}(G)$ : (1)  $\mathcal{M}(H, \theta) \leq (H, \theta)$ , (2)  $\mathcal{M}(H, \theta)$  is an inductive source in  $H$ , and (3) if  $\mathcal{M}(H, \theta) < (H, \theta)$ , then  $I_H(\mathcal{M}(H, \theta)) < H$ . Given a source map  $\mathcal{M}$  on  $G$  and a pair  $(H, \theta) \in \text{pair}(G)$ , we set  $\mathcal{M}(H, \theta) = (M, \mu)$ . Let  $T$  be the stabilizer in  $H$  of  $(M, \mu)$ . Since  $(M, \mu)$  is inductive source for  $H$ , we know that induction is a bijection from  $\text{Irr}(T \mid \mu)$  to  $\text{Irr}(H \mid \mu)$ . Since  $\theta \in \text{Irr}(M \mid \mu)$ , there is a unique character  $\tau \in \text{Irr}(T \mid \mu)$  so that  $\tau^G = \theta$ . Thus, given  $\mathcal{M}$ , we can define a second map  $\mathcal{S}_{\mathcal{M}} : \text{pair}(G) \rightarrow \text{pair}(G)$  by  $\mathcal{S}_{\mathcal{M}}(H, \theta) = (T, \tau)$  as defined above.

In [10], we defined a source map  $\mathcal{Z}$  from  $\text{pair}(G)$  to  $\text{pair}(G)$  as follows. Consider  $(H, \theta) \in \text{pair}(G)$ . If  $\theta_{N \cap H}$  is homogeneous for all  $N \in \mathcal{N}$ , then  $\mathcal{Z}(H, \theta) = (H, \theta)$ . Otherwise, define  $\mathcal{Z}(H, \theta)$  to be the  $\pi$ -factored pair  $(N \cap H, \gamma)$  that is minimal such that  $N \in \mathcal{N}$  and  $\gamma$  is not  $H$ -invariant. We showed in Lemma 4.3 of [10] that  $\mathcal{Z}$  is a  $\pi$ -source map, i.e. that  $\mathcal{Z}(H, \theta) \leq \mathcal{Z}(\mathcal{S}_{\mathcal{Z}}(H, \theta))$  for all  $(H, \theta) \in \text{pair}(G)$ , and that the nucleus determined by  $\mathcal{Z}$  is a self-stabilizing pair.

We will say that a source map  $\mathcal{M}$  is  $\pi$ -closed if  $\mathcal{M}$  is a source map and the following condition holds. If  $(H, \theta) \in \text{pair}(G)$  with  $\mathcal{M}(H, \theta) = (S, \sigma)$  where  $\sigma$  is  $\pi$ -special, then for every  $\pi'$ -special character  $\delta \in \text{Irr}(S)$ , the pair  $(S, \sigma\delta)$  is an inductive source for  $H$ . Note that this is a modification of the definition of  $\pi$ -closed found in [12]. The following theorem is Theorem 2.1 of [12]. It is a generalization of a result of Navarro: Theorem A in [14]. We claim that the proof of this theorem under our modified definition of  $\pi$ -closed is identical to the proof using the definition found in [12].

**Theorem 3.5.** *Let  $\pi$  be a set of primes, let  $G$  be a finite  $\pi$ -separable group, and let  $M$  be a  $\pi$ -closed source map on  $G$  so that  $\mathcal{M}(H, \theta) \leq \mathcal{M}(\mathcal{S}_{\mathcal{M}}(H, \theta))$  for all  $(H, \theta) \in \text{pair}(G)$ . Suppose  $\chi \in B_{\pi}(G : M)$  with  $\mathcal{M}$ -nucleus  $(W, \gamma)$ . Let  $\mathcal{A}$  be the set of  $\pi'$ -special characters of  $W$ . Then the map  $\delta \mapsto (\gamma\delta)^G$  is an injection from  $\mathcal{A}$  to  $\text{Irr}(G)$ .*

We now have the following corollary to this theorem which yields for self-stabilizing pairs a result similar to Navarro's theorem.

**Corollary 3.6.** *Let  $\pi$  be a set of primes, let  $G$  be a  $\pi$ -separable group, and let  $\mathcal{N}$  be a normal  $\pi$ -series for  $G$ . Suppose that  $(V, \alpha)$  is a self-stabilizing pair with respect to  $\mathcal{N}$  where  $\alpha$  is  $\pi$ -special. Then the map  $\beta \mapsto (\alpha\beta)^G$  is an injection from the  $\pi'$ -special characters of  $V$  to  $\text{Irr}(G)$ .*

*Proof.* Let  $\chi = \alpha^G$ . We know that  $(V, \alpha)$  is the  $\mathcal{Z}$ -nucleus of  $\chi$ . The result will follow from Theorem 3.5 if we can show that  $\mathcal{Z}$  is  $\pi$ -closed. Suppose  $(H, \theta) \in \text{pair}(G)$  and  $\mathcal{Z}(H, \theta) = (S, \sigma)$  where  $\sigma$  is  $\pi$ -special. We know that  $S = H \cap N$  for some  $N \in \mathcal{N}$ . This implies that  $S$  is normal in  $H$ . Thus,  $(S, \sigma\delta)$  will be an inductive source for all  $\pi'$ -special  $\delta \in \text{Irr}(S)$ .  $\square$

We can now prove Theorem 2.

*Proof of Theorem 2.* We have  $\varphi \in I_{\pi}(G)$  and  $\chi \in B_{\pi}(G : \mathcal{N})$  so that  $\chi^0 = \varphi$  where  $(V, \gamma)$  is a self-stabilizer pair for  $\chi$ . Since  $\chi \in B_{\pi}(G : \mathcal{N})$ , we know that  $\gamma$  is  $\pi$ -special. By



Corollary 3.3,  $(V, \gamma)$  is inductive. Then applying Lemma 3.4, we deduce that  $(V, \gamma\beta)$  is inductive if  $\beta \in \text{Irr}(V/V')$  with  $\pi'$ -order. Next, we use Corollary 3.6 to see that there is an injection from the characters in  $\text{Irr}(V/V')$  with  $\pi'$ -order to  $\text{Irr}(G)$  by  $\beta \mapsto (\gamma\beta)^G$ . Observe that  $((\gamma\beta)^G)^0 = (\gamma^0\beta^0)^G = (\gamma^0)^G = (\gamma^G)^0 = \chi^0 = \varphi$ . Hence, we have an injection from the characters in  $\text{Irr}(V/V')$  with  $\pi'$ -order to the  $\pi$ -lifts of  $\varphi$ , and this gives the desired conclusion.  $\square$

## 4 An example

In this section, we construct a group  $G$ , and a partial character  $\varphi$  of  $G$ . We will study the  $\pi$ -lifts of  $\varphi$ , and show that they have a number of properties that illustrate the work we have done in this paper. In particular, we will show that the  $\pi$ -lifts of this group are not determined by a single subgroup answering the question of Cossey that was raised in the introduction. We will also show that the number of  $\pi$ -lifts does not divide  $|G|$  which was another question that Cossey has raised. We will show that for different  $\pi$ -normal series  $\mathcal{N}$ , one gets different sets of  $\mathcal{N}$ - $\pi$ -lifts. We will also show that this group has an inductive pair that is not self-stabilizing.

**Construction 4.1.** Let  $E$  be an extra-special group of order  $3^3$  (and exponent 3). Also, take  $V$  to be an elementary abelian group of order  $7^2$ . We write  $V = V_1 \times V_2$  where  $V_1$  and  $V_2$  both have order 7. Let  $M_1$  and  $M_2$  be maximal subgroups of  $E$  so that  $M_1 \neq M_2$ . We define an action of  $E$  on  $V$  so that  $E$  acts on  $V_1$  with  $M_1$  as its kernel and  $E$  acts on  $V_2$  with  $M_2$  as its kernel. Let  $G$  be the semi-direct product of  $E$  acting on  $V$ . Hence,  $|G| = 3^3 \cdot 7^2$ . Now,  $\text{Irr}(G/V) = \text{Irr}(E)$  has 9 linear characters and 2 characters of degree 3. Also,  $\text{Irr}(V) \setminus \{1\}$  has 4 orbits of size 3 (consisting of the characters with either  $V_1$  or  $V_2$  as their kernels) and 4 orbits of size 9 under the action (the characters that do not have either  $V_1$  or  $V_2$  as their kernels). Each of these characters extend to their stabilizers in  $G$  to yield 36 characters of degree 3 and 12 characters of degree 9 in  $\text{Irr}(G)$  that do not have  $E$  in their kernels.

We take  $\pi = \{3\}$ . It is not difficult to see that  $B_\pi(G) = \text{Irr}(G/V)$ . Fix a character  $\chi \in B_\pi(G)$  with  $\chi(1) = 3$ . Let  $\varphi = \chi^0$ .

**Lemma 4.2.** *There are 13 lifts of  $\varphi$  in  $\text{Irr}(G)$ .*

*Proof.* We want to count the number of lifts of  $\varphi$ . Let  $X/V$  be the center of  $G/V$ . We know that  $\chi$  is fully ramified with respect to  $G/X$ , and we take  $\hat{\lambda} \in \text{Irr}(X/V)$  to be the unique irreducible constituent of  $\chi_X$ . It follows that  $\hat{\lambda}^0$  is the unique irreducible constituent of  $\chi_X^0$ . Let  $Z$  be the center of  $G$  and observe that  $Z$  is the center of  $E$ . Notice that  $X = Z \times V$ , and so, restriction is a bijection from  $\text{Irr}(X/V)$  to  $\text{Irr}(Z)$ . We write  $\lambda = \hat{\lambda}_Z$ , and we have  $\hat{\lambda} = \lambda \times 1_V$ . If  $\eta \in \text{Irr}(X)$  with  $\eta^0 = \hat{\lambda}^0$ , then it is not difficult to see that  $\eta = \lambda \times \alpha$  for some character  $\alpha \in \text{Irr}(V)$ .

Suppose  $\psi$  is a lift of  $\varphi$ . Let  $\xi$  be an irreducible constituent of  $\psi_X$ . Now,  $X$  is abelian, so  $\xi$  is linear. This implies that  $\xi^0$  is irreducible. We see that  $\xi^0$  is a constituent of  $\psi_X^0 = \chi_X^0$ . This implies that  $\xi^0 = \hat{\lambda}^0$ , and hence,  $\xi = \lambda \times \alpha$  for some character  $\alpha \in \text{Irr}(V)$ . If  $\alpha = 1_V$ , then we have  $\xi = \hat{\lambda}$ , and hence,  $\psi = \chi$ . Suppose  $\alpha \neq 1_V$ . If  $\alpha$  lies in an orbit of size 9 under

the action of  $E$ , then 9 divides  $\psi(1)$  (since  $\alpha$  is a constituent of  $\psi_V$ ). Since  $\psi(1) = \chi(1) = 3$ , this cannot occur. Hence,  $\alpha$  must lie in an orbit of size 3. This implies that either  $V_1$  or  $V_2$  is in the kernel of  $\alpha$ . Let  $\mathcal{B} = \{\beta \in \text{Irr}(V) \mid \ker(\beta) = V_i \text{ for } i = 1, 2\}$ . We have shown that if  $\psi$  is a lift of  $\varphi$  other than  $\chi$ , then the irreducible constituents of  $\psi_X$  have the form  $\lambda \times \beta$  where  $\beta \in \mathcal{B}$ .

We now show that if  $\psi \in \text{Irr}(G \mid \lambda \times \beta)$  where  $\beta \in \mathcal{B}$ , then  $\psi$  is a lift of  $\varphi$ . Without loss of generality, we may assume that  $\ker(\beta) = V_2$ . Since  $V = V_1 \times V_2$  and  $M_1$  centralizes  $V_1$ , we have that  $M_1$  stabilizes  $\beta$ . Also,  $|G : M_1V| = 3$ , so  $M_1V$  is the stabilizer of  $\beta$  in  $G$ . Since  $V$  is a Hall subgroup, we see that  $\beta$  has a canonical extension  $\hat{\beta} \in \text{Irr}(M_1V)$ . By Clifford's and Gallagher's theorems, we know there is a unique character  $\delta \in \text{Irr}(M_1V/V)$  so that  $(\delta\hat{\beta})^G = \psi$ . We know that  $\delta$  is an extension of  $\hat{\lambda}$ , and since  $G/V$  is extra-special, we know that  $\delta^G = \chi$ . Observe that  $\hat{\beta}^0 = 1$ , so  $\psi^0 = ((\delta\hat{\beta})^G)^0 = (\delta^G)^0 = \chi^0 = \varphi$ .

Notice that  $\mathcal{B}$  contains 12 characters which form 4 orbits of size 3 under the action of  $G$ . Fixing  $\beta \in \mathcal{B}$ , we see that  $\lambda \times \beta$  has 3 extensions to its stabilizer. Thus, each of the orbits of  $\mathcal{B}$  yields 3 lifts of  $\varphi$ . Thus, we obtain in total 13 lifts of  $\varphi$  by including  $\chi$  with the 12 lifts coming from the various orbits on  $\mathcal{B}$ .  $\square$

Notice that this shows that there is no pair  $(U, \delta)$  where  $\delta$  is  $\pi$ -special so that all lifts of  $\varphi$  can be obtained as images of a bijection  $\gamma \mapsto (\delta\gamma)^G$  where  $\gamma$  is a  $\pi'$ -special and linear character of  $U$  since the  $\pi'$ -special, linear characters will form a group isomorphic to a subgroup of  $U/U'$  and hence, the number of such characters would divide  $|U|$  and hence  $|G|$ . But, we showed that  $\varphi$  has 13 lifts and 13 does not divide  $|G|$ .

We now show that we can find an inductive pair that is not self-stabilizing. We set  $\mathcal{N} = \{1, V, G\}$ ,  $\mathcal{N}_1 = \{1, V, M_1V, G\}$ , and  $\mathcal{N}_2 = \{1, V, M_2V, G\}$ . Observe that  $\chi_V = \chi(1)1_G$ , so  $(G, \chi)$  is the self-stabilizing pair for  $\chi$  with respect to  $\mathcal{N}$ . On the other hand, since for  $i = 1$  or  $2$  we know that  $\chi$  is induced from  $\delta_i \in \text{Irr}(M_iV)$  where  $\delta_i$  is an extension of  $\hat{\lambda}$  to  $M_iV$ , it follows that  $(M_iV, \delta_i)$  is the self-stabilizing pair  $\chi$  with respect to  $\mathcal{N}_i$ . This implies that each  $(M_iV, \delta_i)$  is an example of a pair that is inductive for  $\mathcal{N}$  but not a self-stabilizing pair with respect to  $\mathcal{N}$ . Thus, we have found three different pairs  $(G, \chi)$ ,  $(M_1V, \delta_1)$ , and  $(M_2V, \delta_2)$  inducing  $\chi$  that are inductive for  $\mathcal{N}$ . Notice that  $(M_1V, \delta_1)$  is also inductive for  $\mathcal{N}_1$ , but is not inductive for  $\mathcal{N}_2$ . Also,  $(M_iV, \delta_2)$  is inductive for  $\mathcal{N}_2$ , but is not inductive for  $\mathcal{N}_1$ . Thus, the pairs  $(M_iV, \delta_i)$  are inductive for  $\mathcal{N}$ , but not self-stabilizing, and hence, we have our pairs that are inductive but not self-stabilizing.

We now show that  $\chi$  is the only  $\pi$ -lift for  $\varphi$  with the property that for every normal subgroup  $N$  that the irreducible constituents of  $\chi_N$  are  $\pi$ -lifts. Observe that  $M_1V = V_1 \times M_1V_2$ , and so,  $(M_1V)' = V_2$ . It follows that  $|M_1V/(M_1V)'|_{3'} = 7$ . In fact,  $M_1V/(M_1V)' \cong M_1 \times V_1$ . Let  $\mathcal{M}_1$  be the seven characters obtained by  $\zeta \mapsto (\beta_1\zeta)^G$  as  $\zeta$  runs through the linear characters of  $\text{Irr}(M_1V)$  with order 7. In other words,  $\zeta$  runs through the characters in  $\text{Irr}(V_1)$ . We know from Theorem 2 that all of the characters in  $\mathcal{M}_1$  are  $\mathcal{N}_1$ - $\pi$ -lifts. If  $1 \neq \zeta \in \text{Irr}(V_1)$ , then  $M_1V$  is the stabilizer in  $G$  of both  $\zeta$  and  $\delta_1\zeta$ . It follows that  $ZV$  is the stabilizer in  $M_2V$  of  $\hat{\lambda}\zeta$ . In other words,  $(\hat{\lambda}\zeta)^{M_2V}$  is irreducible. Applying Mackey's theorem, the irreducible constituents of  $(\delta_1\zeta)^G|_{M_2V}$  have degree 3 and thus are not  $\pi$ -lifts. Thus,  $\chi$  is the only character in  $\mathcal{M}_1$  that is an  $\mathcal{N}_1$ - $\pi$ -lift.

In a similar manner, we can define  $\mathcal{M}_2$  with respect to  $(M_2V)$ . Working as above, we will see that  $\mathcal{M}_2$  contains 7 characters that are  $\mathcal{N}_2$ - $\pi$ -lifts of  $\varphi$ , and only,  $\chi$  is also an  $\mathcal{N}_1$ - $\pi$ -lift. It

follows that  $\mathcal{M}_1 \cap \mathcal{M}_2 = \{\chi\}$ , and so,  $|\mathcal{M}_1 \cup \mathcal{M}_2| = 13$ . We can conclude that  $\mathcal{M}_1 \cup \mathcal{M}_2$  is the set of all lifts of  $\varphi$ . Notice that this implies that all lifts of  $\varphi$  are  $\mathcal{N}$ - $\pi$ -lifts. In particular, it is possible to have  $\mathcal{N}$ - $\pi$ -lifts that do not arise from a self-stabilizing pair.

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